

# Factorization of correlation functions in coset conformal field theories.

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## Abstract

We use the conformal Ward identities to study the structure of correlation functions in coset conformal field theories. For a large class of primary fields of arbitrary  $g/h$  theory a factorization ansatz is found. Corresponding correlation functions are explicitly expressed in terms of correlation functions of two independent WZNW theories for  $g$  and  $h$ .

Coset theories is an important subclass of two-dimensional conformal invariant QFT's. (For a review see e.g. [1].) The  $g/h$  coset theory is based on the Virasoro algebra generated by [2, 3]

$$K(m) = L^g(m) - L^h(m), \quad m \in Z. \quad (1)$$

The operator  $L^g(m)$  is a conformal generator of the Wess-Zumino-Novikov-Witten (WZNW) theory [4-6] for the Lie algebra  $g$  and  $h \subset g$ . In this paper we study the connection between the coset and WZNW theories which follows from (1) at the level of correlation functions and primary fields.

In the case of  $su(2)/u(1)$  correlation functions of primary fields may be written in terms of correlation functions of the independent WZNW theories

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for  $su(2)$  and  $u(1)$  [7]. Correlation functions of the  $g/u(1)^d$ ,  $d = 1 \dots rank g$ , cosets [8] have a similar structure. In [9] some correlation functions of minimal models were expressed in terms of correlation functions of two WZNW theories.

In this paper we show that a large class of correlation functions of arbitrary  $g/h$  coset conformal field theory can be expressed in terms of correlation functions of two independent WZNW theories for  $g$  and  $h$ . To find correlation functions of coset primary fields we use the conformal Ward identities [10]. We propose an ansatz for coset primary fields and show that the corresponding correlation functions satisfy the Ward identities. Different factorization properties of  $g/h$  coset correlation functions were found in [11].

The results of this paper are in agreement with those of refs. [12, 13] where the  $g/h$  WZNW model was studied by using path integral approach.

We begin with the affine Lie algebra  $\hat{g}_k$  for simple  $g$

$$[J^a(m), J^b(n)] = if^{abc} J^c(m+n) + km\delta^{ab}\delta_{m+n,0},$$

where  $f^{abc}$  are the structure constants of  $g$  and  $k$  is the central charge.

The conformal generator  $L^g(m)$  is given by

$$L^g(m) = \frac{1}{2k + Q_g} \sum_n : J^a(m-n) J^a(n) :,$$

where  $Q_g$  is the quadratic Casimir in the adjoint representation of  $g$ . These operators satisfy the commutator relations

$$[L^g(m), L^g(n)] = (m-n)L^g(m+n) + c^g \left[ \frac{1}{12}(m^3 - m) \right] \delta_{m,-n}, \quad (2)$$

$$c^g = \frac{2k \dim g}{2k + Q_g},$$

where  $c^g$  is the central charge.

Let  $G_R(z)$  be the primary field of  $\hat{g}_k$

$$[J^a(m), G_R(z)] = z^m G_R(z) t_R^a, \quad (3)$$

$$[t_R^a, t_R^b] = if^{abc} t_R^c,$$

where  $t_R^a$  is the representation of the generators of  $g$  for the field  $G_R(z)$ . In the WZNW theory  $G_R(z)$  also is the primary field of the Virasoro algebra (2)

$$[L^g(m), G_R(z)] = z^{m+1} \partial_z G_R(z) + \Delta_R(m+1) z^m G_R(z),$$

$$\Delta_R = \frac{Q_J}{2k + Q_g},$$

where  $Q_R$  is the quadratic Casimir of  $g$  in the representation  $R$ . Here and in what follows we treat only the holomorphic part.

Let  $\hat{h}_k$  be a subalgebra of  $\hat{g}_k$ . We assume that it is generated by  $J^A(m)$ ,  $A = 1 \dots \dim h$ . The field  $G_R$  can be decomposed in the set of some irreducible representations of  $h$

$$G_R(z) = \sum_l G_{Rl}(z) = \sum_l P_l G_R(z), \quad (4)$$

where  $G_{Rl}(z)$  belongs to the  $l$ 's representation and  $P_l$  is the corresponding projector. The field  $G_{Rl}$  satisfies the equation

$$[J^A(m), G_{Rl}(z)] = z^m G_{Rl}(z) t_l^A, \quad (5)$$

where  $t_l^A$  is the representation of the generators of  $h$  for the field  $G_{Rl}(z)$ . As well as  $G_R(z)$  the field  $G_{Rl}(z)$  is the primary field of the Virasoro algebra (2)

$$[L^g(m), G_{Rl}(z)] = z^{m+1} \partial_z G_{Rl}(z) + \Delta_R(m+1) z^m G_{Rl}(z), \quad (6)$$

Correlation functions of these fields can be computed using correlation functions of the WZNW theory

$$\langle G_{R_1 l_1}(z_1) \dots G_{R_N l_N}(z_N) \rangle = \prod_{i=1}^N P_{l_i} \langle G_{R_1}(z_1) \dots G_{R_N}(z_N) \rangle$$

The coset conformal generators  $K(m)$  (1) satisfy commutator relations (2) with the central charge  $c^{g/h} = c^g - c^h$  [3]. We shall need the relation

$$\begin{aligned}
[K(m), G_{Rl}(z)] &= z^{m+1} \left( \partial_z G_{Rl}(z) - \frac{2}{2k + Q_h} : J^A(z) G_{Rl}(z) : t_l^A \right) \\
&+ \Delta_{Rl}(m+1) z^m G_{Rl}(z),
\end{aligned} \tag{7}$$

where

$$: J^A(z) G_{Rl}(z) := \sum_{m < 0} J^A(m) z^{-m-1} G_{Rl}(z) + G_{Rl}(z) \sum_{m \geq 0} J^A(m) z^{-m-1}.$$

$\Delta_{Rl}$  is given by

$$\Delta_{Rl} = \Delta_R - \frac{Q_l}{2k + Q_h}, \tag{8}$$

where  $Q_l$  is the quadratic Casimir of  $h$  in the representation  $l$ .

Correlation functions of the coset primary fields  $\phi_i$  satisfy the conformal Ward identity [10]

$$< K(z) \phi_1(z_1) \dots \phi_N(z_N) > = \sum_{i=1}^N \left\{ \frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right\} < \phi_1(z_1) \dots \phi_N(z_N) >, \tag{9}$$

where  $K(z) = \sum_m K(m) z^{-m-2}$  and  $\Delta_1, \Delta_2, \dots, \Delta_N$  are dimensions of  $\phi_1, \phi_2, \dots, \phi_N$ , respectively.

To find a solution of this equation we shall use an auxiliary WZNW theory. Let  $\hat{h}_{k'}$  be the auxiliary affine Lie algebra

$$[\chi^A(m), \chi^B(n)] = i f^{ABC} \chi^C(m+n) + k' m \delta^{AB} \delta_{m+n,0}. \tag{10}$$

where  $A, B, C = 1 \dots \dim h$ . The value of  $k'$  will be defined later. Let  $\Phi_l$  be the primary field of the WZNW theory for  $\hat{h}_{k'}$

$$[\chi^A(m), \Phi_l(z)] = z^m \Phi_l(z) t_l^{*A}, \tag{11}$$

$$\frac{\partial}{\partial z} \Phi_l(z) = \frac{2}{2k' + Q_h} : \chi^A(z) \Phi_l(z) : t_l^{*A}, \tag{12}$$

where  $t_l^{*A} = -(t_l^A)^T$  and  $Q_h$  is the quadratic Casimir in the adjoint representation of  $h$ . Eq.(12) was introduced in ref. [5].

We look for a solution of eq.(9) in the following factorized form

$$\begin{aligned}
& \sum_{\alpha_1 \dots \alpha_N} < G_{R_1 l_1}^{\alpha_1}(z_1) \dots G_{R_N l_N}^{\alpha_N}(z_N) > < \Phi_{l_1}^{\alpha_1}(z_1) \dots \Phi_{l_N}^{\alpha_N}(z_N) > \\
& \equiv < (G_{R_1 l_1}(z_1), \Phi_{l_1}(z_1)) \dots (G_{R_N l_N}(z_N), \Phi_{l_N}(z_N)) >, \tag{13}
\end{aligned}$$

where  $(\cdot, \cdot)$  is the bilinear form

$$(G_{Rl}(z), \Phi_l(z)) = \sum_{\alpha=1}^{dim l} G_{Rl}^{\alpha}(z) \Phi_l^{\alpha}(z).$$

We shall denote

$$\tilde{G}_{Rl}(z) = (G_{Rl}(z), \Phi_l(z)). \tag{14}$$

It follows from (5) and (11) that  $\tilde{G}_{Rl}(z)$  commutes with the operator  $\tilde{J}^A(m) = J^A(m) + \chi^A(m)$

$$[\tilde{J}^A(m), \tilde{G}_{Rl}(z)] = 0. \tag{15}$$

The vacuum state  $|0\rangle$  is the joint state of the  $\hat{g}_k$  and  $\hat{h}_{k'}$  WZNW theories. We shall use the following properties of  $|0\rangle$

$$< 0 | \tilde{J}^A(m < 0) = \tilde{J}^A(m \geq 0) | 0 > = 0, \tag{16}$$

$$< 0 | K_-(z) = K_+(z) | 0 > = 0, \tag{17}$$

where

$$K_-(z) = \sum_{m < -1} K(m) z^{-m-2}, \quad K_+(z) = \sum_{m \geq -1} K(m) z^{-m-2}.$$

Let us compute the left-hand side of eq.(9) using correlation functions (13). To simplify presentation we assume that  $|z| > |z_1| > \dots > |z_N|$ . Using eqs.(17), (7) and (12) we get

$$\begin{aligned}
& < K(z) \tilde{G}_{R_1 l_1}(z_1) \dots \tilde{G}_{R_N l_N}(z_N) > = < K_+(z) \tilde{G}_{R_1 l_1}(z_1) \dots \tilde{G}_{R_N l_N}(z_N) > \\
& = \left\{ \frac{\Delta_{R_1 l_1}}{(z - z_1)^2} + \frac{1}{z - z_1} \frac{\partial}{\partial z_1} \right\} < \tilde{G}_{R_1 l_1}(z_1) \dots \tilde{G}_{R_N l_N}(z_N) > \\
& + < \tilde{G}_{R_1 l_1}(z_1) K_+(z) \dots \tilde{G}_{R_N l_N}(z_N) > + \frac{1}{z - z_1} < T_{R_1 l_1}(z_1) \dots \tilde{G}_{R_N l_N}(z_N) >, (18)
\end{aligned}$$

where

$$T_{R_1 l_1}(z) = \frac{1}{2k' + Q_h} (G_{R_1 l_1}(z) t^A, : \chi^A(z) \Phi_{l_1}(z) :) - \frac{1}{2k + Q_h} (: J^A(z) G_{R_1 l_1}(z) : t^A, \Phi_{l_1}(z)).$$

At  $k' = -k - Q_h$  the field  $T_{R_1 l_1}(z)$  can be written in the form

$$T_{R_1 l_1}(z) = -\frac{1}{2k + Q_h} : \tilde{J}^A(z) (G_{R_1 l_1}(z) t_{l_1}^A, \Phi_{l_1}(z)) :.$$

Due to eqs. (15) and (16) the last term of eq.(18) vanishes

$$< T_{R_1 l_1}(z_1) \dots \tilde{G}_{R_N l_N}(z_N) > = 0.$$

Proceeding inductively one can show that the correlation function (13) satisfies the Ward identity (9).

From the arguments presented above it follows that  $\tilde{G}_{Rl}(z)$  (14) represents the primary field of the  $g/h$  coset theory which has the conformal dimension (8). We took the fields  $G_{Rl}(z)$  from the decomposition (4). However to prove the factorization only eqs. (5) and (6) were essentially used. These equations have other solutions which can be used to construct coset primary fields.

To construct coset currents let us consider the field  $J(z) = (J^i(z))$ ,  $J^i(z) = \sum_m J_m^i z^{-m-1}$ ,  $i = \dim h + 1 \dots \dim g$ . It can be decomposed in the set of some irreducible representations of  $h$

$$J(z) = \sum_s J_s(z).$$

The field  $J_s(z)$  satisfies eq.(5) for some  $t_s^A$  and eq.(6) with the conformal dimension  $\Delta = 1$ . According to eq. (14) the coset current corresponding to  $J_s(z)$  is given by

$$\tilde{J}_s(z) = (J_s(z), \Phi_s(z)). \quad (19)$$

It follows from (8) that  $\tilde{J}_s(z)$  has the conformal dimension

$$1 - \frac{Q_s}{2k + Q_h},$$

where  $Q_s$  is the quadratic Casimir of  $h$  in the representation  $s$ .

Let us consider the  $g/u(1)^d$ ,  $1 \leq d \leq \text{rank } g$ , coset theory. In this case the primary field  $G_R(z)$  is decomposed in the set of one-dimensional representations of  $u(1)^d$

$$G_R(z) = \sum_{\mu=1}^{\dim R} G_{R\mu}(z),$$

$$[J^A(m), G_{R\mu}(z)] = \mu^A z^m G_{R\mu}(z),$$

where  $\mu = (\mu^A)$ . A solution of eqs.(11),( 12) is given by

$$\Phi_\mu(z) =: \exp\left(-\frac{i}{k'} \mu \cdot \varphi(z)\right) :, \quad (20)$$

$$\varphi^A(z) = q^A - i\chi^A(0)\log z + i \sum_{n \neq 0} \frac{\chi^A(n)}{n} z^{-n}, \quad (21)$$

where

$$[q^A, \chi^B(m)] = i\delta^{AB}\delta_{m,0}. \quad (22)$$

According to eqs.(14) and (8) at  $k' = -k$   $\tilde{G}_{R\mu}(z) = G_{R\mu}(z)\Phi_\mu(z)$  represents the coset primary field which has the dimension  $\Delta_{R\mu} = \Delta_R - \mu^2/2k$ .

The correlation function of these fields is given by

$$\langle G_{R_1\mu_1}(z_1) \dots G_{R_N\mu_N}(z_N) \rangle = \langle G_{R_1}(z_1) \dots G_{R_N}(z_N) \rangle \prod_{i < j}^N (z_i - z_j)^{-\frac{\mu_i \mu_j}{k}}.$$

This is in agreement with the results of refs.[7, 8]. Parafermion  $g/u(1)^d$  currents in the form (19) were obtained in [14].

The results presented in this paper can be extended in many directions. The most important is to study the factorization properties of the  $W/h$  coset conformal field theory. It is also interesting to find primary fields and describe the corresponding operator algebra. This is presently being studied.

## References

- [1] M.B.Halpern et al., *Phys.Rept.* **265** (1996) 1.
- [2] K.Bardakci and M.B.Halpern, *Phys.Rev.* **D3** (1971) 2493.  
M.B.Halpern, *Phys.Rev.* **D4** (1971) 2398.
- [3] P.Goddard, A.Kent and D.Olive, *Phys.Lett.* **152B** (1985) 88.
- [4] E.Witten, *Comm.Math.Phys.* **92** (1984) 455.
- [5] V.Knizhnik and A.B.Zamolodchikov, *Nucl.Phys.* **B247** (1984) 83.
- [6] D.Gepner and E.Witten, *Nucl.Phys.* **B278** (1986) 493.
- [7] V.A.Fateev and A.B.Zamolodchikov, *Sov.Phys.JETP* **82** (1985) 215.
- [8] A.V.Bratchikov, Generalized abelian coset conformal field theories. hep-th/9912112.
- [9] I.I.Kogan, A. Lewis and O.A. Soloviev, *Int.J.Mod.Phys.* **A13** (1998) 1345. hep-th/9703028.
- [10] A.A.Belavin, A.M.Polyakov and A.B.Zamolodchikov, *Nucl.Phys.* **B241** (1984) 333.
- [11] M.B.Halpern and N.O.Obers, *Int.J.Mod.Phys.* **A9** (1994) 265. hep-th/9207071.
- [12] K.Gawedzki and A.Kupiainen, *Nucl.Phys.* **B320** (1989) 625.
- [13] D.Karabali, et al. *Phys.Lett.* **216B** (1989) 307.



- [14] A.V.Bratchikov,  $g/u(1)^d$  parafermions from constrained WZNW theories. hep-th/9712243.